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This thesis presents an introduction to the theory of knots and knot groups. The development begins with the definitions of knot, equivalent knots, and tame knots, and progresses to the presentation of the fundamental group of a topological space. The procedure for calculating a knot group is outlined and examples are given utilizing common knots. Two methods for dealing with more complicated knots are introduced. Examples of both methods are given.

AN INTRODUCTION TO KNOTS
AND KNOT GROUPS

by

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TABLE OF CONTENTS

Part	Page
INTRODUCTION.	vi
CHAPTER I. KNOTS AND THE FUNDAMENTAL GROUP	1
CHAPTER II. KNOT GROUPS.	8
CHAPTER III. TECHNIQUES FOR MORE COMPLICATED KNOTS	15
SUMMARY	35
BIBLIOGRAPHY.	36

LIST OF FIGURES

Figure	Page
1. THE RELATION AT A CROSSING.	9
2. THE TREFOIL OR CLOVER-LEAF KNOT	10
3. THE FIGURE EIGHT KNOT	11
4. THE SQUARE KNOT	12
5. THE GRANNY KNOT	13
6. SECTION OF A TORI SEQUENCE FOR K	20
7. A S-SEQUENCE OF 2-SPHERES FOR K	21
8. CONSTRUCTION OF KNOT K_1	24
9. THE 2-SPHERES FOR K_1	24
10. CONSTRUCTION OF K_2	25
11. THE 2-SPHERES FOR K_2	26
12. TORI SEQUENCE FOR K_1	27
13. TORI SEQUENCE FOR K_2	28
14. HOMOMORPHISMS FOR THE VAN KAMPEN THEOREM.	29
15. THE HOMOTOPY TYPE OF X_1	33

INTRODUCTION

The purpose of this paper is to present an introduction to the theory of knots and knot groups assuming an intermediate knowledge of group theory and topology on the part of the reader. Thus Chapter I is concerned with knots, equivalent knots, and tame knots. The basic definitions are given to develop the concept of a fundamental group for a topological space. The trivial knot is defined.

Chapter II defines knot group and calculates the knot group for the trefoil, figure eight and square knots. The chapter ends with a proof of the existence of nontrivial knots.

Chapter III presents two sophisticated methods of calculating knot groups of compound knots. Examples are given to illustrate the process.

The development of knots and the fundamental group in Chapter I is based on material from Crowell and Fox [1]. Chapter II is based primarily on material from Crowell and Fox [1] and Fox [2]. The material on tori sequences and 2-sphere sequences in Chapter III is based heavily on a paper by E. E. Posey [4]. The remainder of the chapter, devoted to the van Kampen Theorem is based on material from Crowell and Fox [1]. The books of Hall [3] and Rotman [5] are recommended as references on group theory. For additional information on homotopy theory consult the book by Whitehead [6].

CHAPTER I

KNOTS AND THE FUNDAMENTAL GROUP

Definition 1.1: A subset K of 3-space E^3 is a knot if there exists a homeomorphism of the unit circle into E^3 whose image is K .

Definition 1.2: Knots K_1 and K_2 are equivalent if there exists a space homeomorphism h (a homeomorphism of E^3 onto E^3) such that $h(K_1) = K_2$.

Clearly, the unit circle itself is a knot. It is often called the trivial or unknotted knot. We assume for the present that there exist nontrivial knots. Since all knots are homeomorphic images of the unit circle and hence homeomorphic to each other, to distinguish between the various knots we must consider space homeomorphisms. The proof that the relation of knot equivalence is a true equivalence relation is straightforward and omitted. Each equivalence class of knots is called a knot type and its members are said to be of the same type. Those knots equivalent to the unit circle constitute the trivial type.

Definition 1.3: A knot which is the finite union of closed straight-line segments is called a polygonal knot.

Definition 1.4: A knot is tame if it is equivalent to a polygonal knot, otherwise the knot is wild. (For this paper we restrict ourselves to tame knots.)

The central theme of knot theory is the study of the fundamental group of the complementary space $E^3 - K$ of a knot K . The rest of this chapter is devoted to a brief discussion of the fundamental group of a topological space.

Definition 1.5: A path in a topological space X is a continuous mapping of the closed unit interval I into X . A path whose initial and terminal points coincide is called a loop, its common endpoint is its basepoint. We let $C(X, x_0)$ denote the set of all loops in X with basepoint x_0 .

Definition 1.6: Two paths p_1 and p_2 of $C(X, x_0)$ are equivalent (denoted $p_1 \sim p_2$) if p_1 is homotopic to p_2 , i.e., if there exists a continuous $H : I \times I \rightarrow X$ such that $H(x, 0) = p_1(x)$ and $H(x, 1) = p_2(x)$ for each x in X .

Theorem 1.1: The relation \sim is a true equivalence relation on the set of all paths in a space X .

Proof: For any path p we have $p \sim p$ since we can define $H(x, y) = p(x)$ for x in X and $0 \leq y \leq 1$. Hence \sim is reflexive. Suppose $p_1 \sim p_2$, then there exists a continuous $F : I \times I \rightarrow X$ such that $F(x, 0) = p_1(x)$ and $F(x, 1) = p_2(x)$ for each x in X . Define $h : I \times I \rightarrow I \times I$ by $h(x, y) = (x, 1-y)$. Clearly h is continuous. Define $G : I \times I \rightarrow X$ by $G = F \circ h$. Then G is continuous and $G(x, 0) = F \circ h(x, 0) = F(x, 1) = p_2(x)$ and $G(x, 1) = F \circ h(x, 1) = F(x, 0) = p_1(x)$. Hence $p_2 \sim p_1$ and \sim is symmetric. Suppose $p_1 \sim p_2$ and $p_2 \sim p_3$. Then there exist continuous F and G from $I \times I$ to X such that

$F(x,0) = p_1(x)$, $F(x,1) = p_2(x)$, $G(x,0) = p_2(x)$ and $G(x,1) = p_3(x)$.

Define $H : I \times I \rightarrow X$ by

$$H(x,y) = \begin{cases} F(x,2y) & \text{if } 0 \leq y \leq 1/2 \\ G(x,2y-1) & 1/2 \leq y \leq 1 \end{cases}.$$

It is obvious that H is continuous and $H(x,0) = p_1(x)$ and

$H(x,1) = p_3(x)$. Hence $p_1 \sim p_3$ and \sim is transitive.

We let $[f]$ denote the equivalence class containing the path f . It can be shown that any member of an equivalence class may be used to denote the class.

Definition 1.7: Let f and g be paths in X . Let $f * g$ be the function defined by

$$f * g(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ g(2x-1) & 1/2 \leq x \leq 1 \end{cases}.$$

Clearly $f * g$ is single-valued and continuous (defined) only if the terminal point of f is the initial point of g . When $f * g$ is defined it is a path in X .

Definition 1.8: If f and g are paths in X , we define $[f] \cdot [g] = [f * g]$.

Theorem 1.2: Let $\pi(X, x_0)$ denote the set of all equivalence classes determined by \sim on the subset $C(X, x_0)$ of the set of all paths in X . (See Definition 1.5.) Then $\pi(X, x_0)$ together with \cdot is a group. It is called the fundamental group of X relative to basepoint x_0 .

Proof: First we show that our operation is well defined. Let f, g, j , and k be in $C(X, x_0)$ such that $f \sim g$ and $j \sim k$. We

must show $f * j \sim g * k$. There exist continuous F and G from $I \times I$ to X such that $F(x,0) = f(x)$, $F(x,1) = g(x)$, $G(x,0) = j(x)$ and $G(x,1) = k(x)$ for each x in I . Define $H : I \times I \rightarrow X$ by

$$H(x,y) = \begin{cases} F(2x,y) & \text{if } 0 \leq x \leq 1/2 \\ G(2x-1,y) & 1/2 \leq x \leq 1 \end{cases}.$$

The function H is continuous since $H(1/2,y) = F(1,y) = x_0 = G(0,y)$.

Also

$$\begin{aligned} H(x,0) &= \begin{cases} F(2x,0) & \text{if } 0 \leq x \leq 1/2 \\ G(2x-1,0) & 1/2 \leq x \leq 1 \end{cases} = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ j(2x-1) & 1/2 \leq x \leq 1 \end{cases} = \\ f * j(x) \text{ and } H(x,1) &= \begin{cases} F(2x,1) & \text{if } 0 \leq x \leq 1/2 \\ G(2x-1,1) & 1/2 \leq x \leq 1 \end{cases} = \\ \begin{cases} g(2x) & \text{if } 0 \leq x \leq 1/2 \\ k(2x-1) & 1/2 \leq x \leq 1 \end{cases} &= g * k(x). \text{ Hence } f * j \sim g * k. \end{aligned}$$

It follows now that $[f] \cdot [j] = [g] \cdot [k]$. Now we show the existence of an identity element. Let $[e]$ denote the equivalence class containing the constant function $e(x) = x_0$ for every x . We must show $[f] \cdot [e] = [f]$ for every f in $C(X, x_0)$. Define

$H : I \times I \rightarrow X$ by

$$H(x,y) = \begin{cases} f(2x/(1+y)) & \text{if } 0 \leq x \leq (1+y)/2 \\ f(1) & (1+y)/2 \leq x \leq 1 \end{cases}.$$

The proof that H is continuous is omitted but may be found in [6].

For the function H we have

$$H(x,0) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ x_0 & 1/2 \leq x \leq 1 \end{cases} = f * e(x) \text{ and}$$

$$H(x,1) = \begin{cases} f(x) & \text{if } 0 \leq x \leq 1 \\ f(1) & x = 1 \end{cases} = f(x).$$

So $f * e \sim f$ and $[f] \cdot [e] = [f * e] = [f]$. We prove now the existence of inverses. For each f in $C(X, x_0)$ define $f^{-1}(x) = f(1 - x)$. We must show $[f] \cdot [f^{-1}] = [e]$. Define $G : I \times I \rightarrow X$ by

$$G(x,y) = \begin{cases} f(0) & \text{if } 0 \leq x \leq y/2 \\ f(2x-y) & y/2 \leq x \leq 1/2 \\ f(2-2x-y) & 1/2 \leq x \leq (2-y)/2 \\ f(0) & (2-y)/2 \leq x \leq 1 \end{cases}.$$

Again the proof that G is continuous may be found in [6]. Since

$$G(x,0) = \begin{cases} f(0) & \text{if } 0 \leq x \leq 0 \\ f(2x) & 0 \leq x \leq 1/2 \\ f(2-2x) & 1/2 \leq x \leq 1 \\ f(0) & 1 \leq x \leq 1 \end{cases} = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ f(2-2x) & 1/2 \leq x \leq 1 \end{cases} =$$

$$\begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ f(1-(2x-1)) & 1/2 \leq x \leq 1 \end{cases} = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ f^{-1}(2x-1) & 1/2 \leq x \leq 1 \end{cases} =$$

$f * f^{-1}(x)$ and

$$G(x,1) = \begin{cases} f(0) & \text{if } 0 \leq x \leq 1/2 \\ f(2x-1) & 1/2 \leq x \leq 1/2 \\ f(2-2x-1) & 1/2 \leq x \leq 1/2 \\ f(0) & 1/2 \leq x \leq 1 \end{cases} = f(0) = x_0,$$

we know $f * f^{-1} \sim e$. Hence $[f] \cdot [f^{-1}] = [f * f^{-1}] = [e]$.

Finally we show the associative property. Let f, g and h be in $C(X, x_0)$. Define $F : I \times I \rightarrow X$ by

$$F(x,y) = \begin{cases} f(4x/(1+y)) & \text{if } 0 \leq x \leq (1+y)/4 \\ g(4x-y-1) & (1+y)/4 \leq x \leq (2+y)/4 \\ h((4x-y-2)/(2-y)) & (2+y)/4 \leq x \leq 1 \end{cases}.$$

Once again the proof that F is continuous may be found in [6].

Now we see that

$$F(x,0) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ g(4x-1) & 1/4 \leq x \leq 1/2 \\ h((4x-2)/2) & 1/2 \leq x \leq 1 \end{cases} =$$

$$\begin{cases} f * g(2x) & \text{if } 0 \leq x \leq 1/2 \\ h(2x-1) & 1/2 \leq x \leq 1 \end{cases} = (f * g) * h(x) \quad \text{and}$$

$$F(x,1) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ g(4x-2) & 1/2 \leq x \leq 3/4 \\ h(4x-3) & 3/4 \leq x \leq 1 \end{cases} =$$

$$\begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2 \\ g * h(2x-1) & 1/2 \leq x \leq 1 \end{cases} = f * (g * h)(x).$$

So $(f * g) * h \sim f * (g * h)$. Now $([f] \cdot [g]) \cdot [h] =$
 $[f * g] \cdot [h] = [(f * g) * h] = [f * (g * h)] = [f] \cdot [g * h] =$
 $[f] \cdot ([g] \cdot [h]).$

The fundamental group of a space X depends on the choice of basepoint. We will now show that in a pathwise connected space the fundamental groups of X for the different basepoints are all isomorphic. A space X is pathwise connected if each pair of its points can be joined by a path.

Theorem 1.3: If X is pathwise connected then $(\pi(X, x_1), \cdot)$ is isomorphic to $(\pi(X, x_2), \cdot)$.

Proof: Let a be a path in X with initial point x_1 and terminal point x_2 . Then a^{-1} has initial point x_2 and terminal point x_1 . Define $G : (\pi(X, x_1), \cdot) \rightarrow (\pi(X, x_2), \cdot)$ by $G([f]) = [a^{-1}] \cdot [f] \cdot [a]$ for each $[f]$ in $\pi(X, x_1)$. It is clear that $[a^{-1}] \cdot [f] \cdot [a]$ is defined and is in $\pi(X, x_2)$. For any $[f]$ and $[g]$ in $\pi(X, x_1)$, $G([f] \cdot [g]) = [a^{-1}] \cdot ([f] \cdot [g]) \cdot [a] = [a^{-1}] \cdot [f * g] \cdot [a] = [a^{-1} * f * g * a] = [a^{-1} * f * a * a^{-1} * g * a] = [a^{-1} * f * a] \cdot [a^{-1} * g * a] = ([a^{-1}] \cdot [f] \cdot [a]) \cdot ([a^{-1}] \cdot [g] \cdot [a])$. Hence G is a homeomorphism. Next, let $[a^{-1}] \cdot [h] \cdot [a] = [e]$. Then $[h] = [a] \cdot [a^{-1}] \cdot [h] \cdot [a] \cdot [a^{-1}] = [a] \cdot [a^{-1}] = [e]$. So the function G is one-to-one. For any $[g]$ in $\pi(X, x_2)$, $[a] \cdot [g] \cdot [a^{-1}]$ is in $\pi(X, x_1)$. Since $G([a] \cdot [g] \cdot [a^{-1}]) = [a^{-1}] \cdot [a] \cdot [g] \cdot [a^{-1}] \cdot [a] = [g]$, the mapping G is onto. Therefore G is an isomorphism.

We close this chapter by stating a fact important to the study of knots. Due to its length, the proof is omitted but may be found in [1].

Theorem 1.4: The fundamental group of the circle is infinite cyclic.

CHAPTER II

KNOT GROUPS

If K is a knot in E^3 then the fundamental group of $E^3 - K$ is called the knot group of K . Since $E^3 - K$ is pathwise connected Theorem 1.3 applies. Hence we will omit reference to any particular basepoint x_0 . There are several ways to calculate the knot groups. For the purpose of this paper, we restrict ourselves to the method of "over presentations." A detailed explanation of over presentations may be found in [1]. We describe briefly the process as used here.

Any simple closed polygon K can be projected by a projection p into the xy -plane in such a manner that for each x on $p(K)$ $p^{-1}(x)$ is a singleton set or has cardinal 2. Further, if x is a vertex of $p(K)$ then $p^{-1}(x)$ is a singleton set. Last, if $p^{-1}(x)$ has cardinal 2 then for all y in $p(K)$ sufficiently near x $p^{-1}(y)$ is a singleton set. A projection with these properties is called a regular projection. We now describe how to read the set of generators and defining relations for a knot group G from the regular projection of a knot K .

One of the two directions along the knot K is chosen to be positive. There are a finite number n of double points which divide K into n arcs. Let x_i , $1 \leq i \leq n$, denote the element of G represented by a loop from a fixed point x which passes

under the i^{th} arc in a left to right direction. Obviously x_i^{-1} is a loop in the opposite direction. It is clear that x_1, \dots, x_n generate G . At each crossing or double point we read off a relation; $x_i x_h^{-1} x_i^{-1} x_k = 1$ if the under arcs are oriented or directed from left to right or $x_i x_h x_i^{-1} x_k^{-1} = 1$ if the under arcs are oriented from right to left.

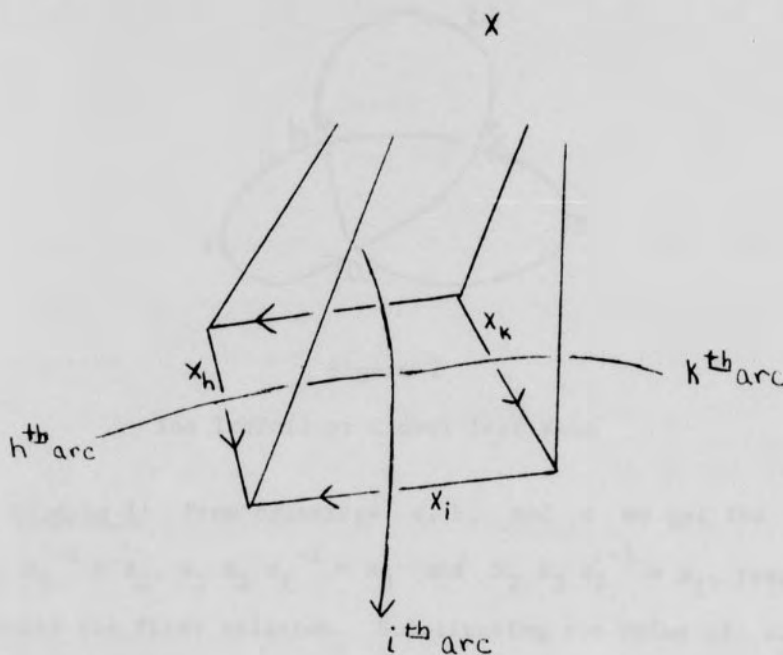


Figure 1

The Relation at a Crossing

However, in both cases the relations reduce to $x_i x_h x_i^{-1} = x_k$. Hence the relation depends only on the orientation of the i^{th} arc, the orientations of the h^{th} and k^{th} arcs are not important. The

n relations obtained in this manner form the system of defining relations for G . It can be shown [1] that any one of the relations is a consequence of the other $n - 1$ relations. Hence our presentation of G will consist of n elements and the $n - 1$ relations, denoted $G = \langle x_1, \dots, x_n : r_1, \dots, r_{n-1} \rangle$. The following examples illustrate the process described above.

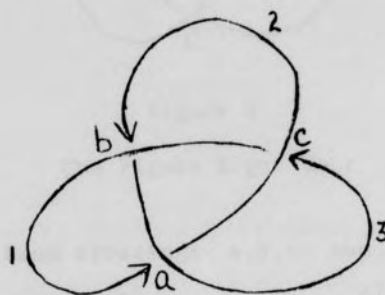


Figure 2

The Trefoil or Clover-leaf Knot

Example 1: From crossings a , b , and c we get the relations $x_3 x_1 x_3^{-1} = x_2$, $x_1 x_2 x_1^{-1} = x_3$ and $x_2 x_3 x_2^{-1} = x_1$, respectively. We delete the first relation. Substituting the value of x_3 into the last relation yields $x_2(x_1 x_2 x_1^{-1})x_2^{-1} = x_1$ or $x_2 x_1 x_2 = x_1 x_2 x_1$. Hence our group presentation is $G = \langle x_1, x_2 : x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$.

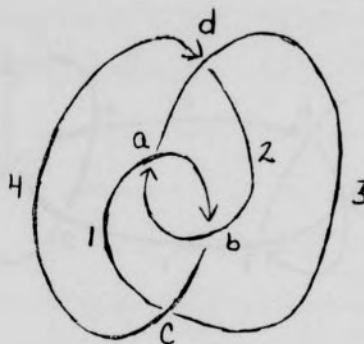


Figure 3

The Figure Eight Knot

Example 2: From crossings a, b, c and d we get the relations $x_1 x_2 x_1^{-1} = x_3$, $x_2 x_1 x_2^{-1} = x_4$, $x_4 x_1 x_4^{-1} = x_3$ and $x_3 x_2 x_3^{-1} = x_4$, respectively. We work with the first three relations only. In the relation from crossing a we substitute the relation from crossing c to obtain $x_1 x_2 x_1^{-1} = x_4 x_1 x_4^{-1}$. We now substitute for x_4 the relation obtained from crossing b . This yields the relation $x_1 x_2 x_1^{-1} = x_2 x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1}$. Finally, we multiply on the right by $x_2 x_1$ and the relation becomes $x_1 x_2 x_1^{-1} x_2 x_1 = x_2 x_1 x_2^{-1} x_1 x_2$. Hence the group presentation is $G = (x_1, x_2 : x_1 x_2 x_1^{-1} x_2 x_1 = x_2 x_1 x_2^{-1} x_1 x_2)$.

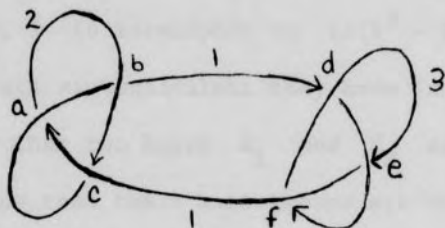


Figure 4

The Square Knot

Example 3: It is geometrically clear that two of the arcs may be identified. Consequently, we label both arcs with the number one. For our presentation we omit the relations from crossings a and e. The remaining crossings b, c, d and f yield $x_1 x_2 x_1^{-1} = x_2$, $x_1 x_2 x_1^{-1} = x_1$, $x_1 x_3 x_1^{-1} = x_3$ and $x_1 x_3 x_1^{-1} = x_1$, respectively. Taking the relation from b we multiply on the left by $x_2^{-1} x_1^{-1}$ and then apply inverses to get $x_1 = x_2^{-1} x_1 x_2$. Using this relation and the relation from c we get $x_1 x_2 x_1^{-1} = x_2^{-1} x_1 x_2$. We first multiply on the left by x_2 and then on the right by x_1 to simplify the relation to $x_2 x_1 x_2 = x_1 x_2 x_1$. Using the relations from crossings d and f and the same process as above gives $x_1 x_3 x_1 = x_3 x_1 x_3$ for the second relation. Hence the group presentation is $G = (x_1, x_2, x_3 : x_1 x_2 x_1 = x_2 x_1 x_2, x_1 x_3 x_1 = x_3 x_1 x_3)$.

We remark that if knots K_1 and K_2 are equivalent, then the complementary spaces $E^3 - K_1$ and $E^3 - K_2$ are homeomorphic and $(\pi(E^3 - K_1), \cdot)$ is isomorphic to $(\pi(E^3 - K_2), \cdot)$. In other words, if two knots are equivalent they have isomorphic knot groups. In order to show that two knots K_1 and K_2 are distinct, it is sufficient to show that their knot groups are not isomorphic. However, it is possible to have isomorphic knot groups but two distinct knots. Using more sophisticated techniques (see the reference in [1] on page 131), it can be shown that the granny knot

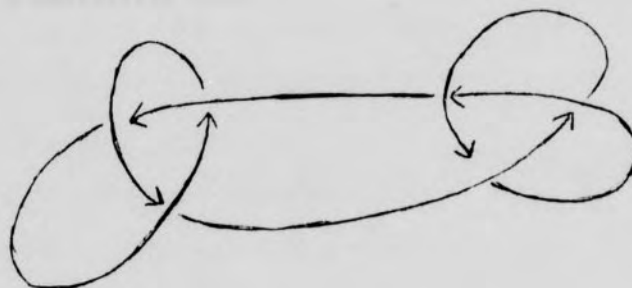


Figure 5

The Granny Knot

and the square knot represent distinct knot types. However, the group presentation of the granny knot is $G = \langle x, y, z : x z x = z x z, y z y = z y z \rangle$ which is clearly isomorphic to the knot group of the square knot.

We conclude this chapter by proving the existence of nontrivial knot types. Consider the symmetric group S_3 generated by the cycles

(12) and (23) . The group S_3 is not abelian since $(12)(23) = (132)$ but $(23)(12) = (123)$. From example one we know the group of the clover-leaf knot is $G = \langle x_1, x_2 : x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$. Define a function h from G to S_3 by $h(x_1) = (12)$ and $h(x_2) = (23)$. Since $h(x_1 x_2 x_1) = h(x_1) h(x_2) h(x_1) = (12)(23)(12) = (13)$ and $h(x_2 x_1 x_2) = h(x_2) h(x_1) h(x_2) = (23)(12)(23) = (13)$, we know that the knot group G is homomorphic to S_3 , a nonabelian group. Hence the knot group of the clover-leaf knot is nonabelian and not isomorphic to the infinite cyclic group of the unit circle. So the clover-leaf knot is an example of a nontrivial knot.

CHAPTER III

TECHNIQUES FOR MORE COMPLICATED KNOTS

The method used in Chapter II is sufficient as long as there are just a few crossings. However, the number of crossings quickly becomes unwieldy. Also, our method is clumsy when dealing with several small knots tied into one large knot. This chapter discusses two ways to find the knot group of large complicated knots. We begin with a definition from [5].

Definition 3.1: Let $\{A_\alpha\}$, α a member of some index set I , be a family of groups. A free product of the A_α is a group P having the following properties:

- (i) P contains an isomorphic copy of each A_α , i.e., for each α there is a homomorphism $i_\alpha : A_\alpha \rightarrow P$ that is one-to-one.
- (ii) For every group G and every family of homomorphisms $f_\alpha : A_\alpha \rightarrow G$, α a member of I , there is a unique homomorphism $\psi : P \rightarrow G$ extending each f_α , i.e., for each α , $\psi i_\alpha = f_\alpha$.

Theorem 3.1: If $\{A_\alpha\}$, α a member of I , is a family of groups, a free product of the A_α does exist.

Proof: We may assume that the A_α are pairwise disjoint. We call $\bigcup_\alpha A_\alpha$ the alphabet and the elements are letters. A word w is reduced if $w = e_\alpha$, where e_α is the identity of A_α , or

if $w = a_1 a_2 \cdots a_k$ where no $a_i = e_\alpha$ and no adjacent letters lie in the same A_α . For all words, reduced or otherwise, we define the following elementary equivalence operations:

(a) $a_1 a_2 \cdots a_{i-1} a_i a_{i+1} \cdots a_t$ is equivalent to

$a_1 \cdots a_{i-1} a_{i+1} \cdots a_t$ if a_i is the identity of some A_α ,

(b) $a_1 a_2 \cdots a_{i-1} a_i a_{i+1} \cdots a_t$ is equivalent to

$a_1 a_2 \cdots a_{i-1} a_i^* a_{i+2} \cdots a_t$ if a_i and a_{i+1} belong to the same group A_α and $a_i a_{i+1} = a_i^*$ in A_α . We say that two words w_1

and w_2 are equivalent if there is a finite sequence such that

$w_1 = y_1, y_2, \cdots, y_n = w_2$ with y_i and y_{i+1} elementary equivalent of type (a) or (b) for $i = 1, 2, \dots, n-1$. This equivalence relation

divides the words into classes. It can be shown [3] that there is a

unique reduced word in each class. We define the product of two

classes as $[w_1][w_2] = [w_1 w_2]$ which can be shown to be independent

of the representatives chosen for each class. The product is associ-

ative and with the void words as identity forms a group G . We now

show that G has the properties mentioned in Definition 3.1.

Consider A_α . Each distinct element a_j of A_α is a reduced word.

Define a mapping $i_\alpha : A_\alpha \rightarrow P$ by $i_\alpha(a_j) = [a_j]$ for each a_j in

A_α . Clearly i is an isomorphism so property (i) is satisfied. Now

let H be a group and $\{f_\alpha : A_\alpha \rightarrow H, \alpha \text{ a member of } I\}$ be a family

of homomorphisms. We define $\psi : G \rightarrow H$ in the following manner.

Pick $[x]$ a member of G . Then $[x] = [a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_k}]$ where

$a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_k}$ is a reduced word and a_{α_j} is a member of A_{α_j} .

Then $\psi([x]) = f_{\alpha_1}(a_{\alpha_1})f_{\alpha_2}(a_{\alpha_2}) \cdots f_{\alpha_k}(a_{\alpha_k})$. Property (ii)

follows immediately.

Theorem 3.2: Let $\{A_\alpha\}$ be a family of groups, and let G and H each be free products of the A_α . Then G is isomorphic to H .

Proof: Let $i_\alpha : A_\alpha \rightarrow G$ and $j_\alpha : A_\alpha \rightarrow H$ be the embeddings and let $h_\alpha : A_\alpha \rightarrow A_\alpha$ be the identity on A_α . Then $f_\alpha = j_\alpha h_\alpha : A_\alpha \rightarrow H$, so that there is a map $\psi : G \rightarrow H$ extending each f_α . Similarly, if $g_\alpha = i_\alpha h_\alpha : A_\alpha \rightarrow G$, there is a map $\theta : H \rightarrow G$ extending each g_α . The composite $\theta\psi : G \rightarrow G$ is such that $\theta\psi i_\alpha = i_\alpha$ for every α . Since the identity map $I : G \rightarrow G$ also has the property that $I i_\alpha = i_\alpha$ for all α , the uniqueness part of the definition of free product implies that $\theta\psi = I$. Similarly, $\psi\theta$ is the identity on H , so that ψ is an isomorphism [5].

We denote the free product of the A_α by $\bigotimes A_\alpha$. If the index set is finite, we sometimes denote $\bigotimes A_\alpha$ by $A_1 \otimes A_2 \otimes \cdots \otimes A_n$.

Definition 3.2: Let A_α , $\alpha \in I$, be a set of groups indexed by the set I . Let us suppose that each A_α contains a subgroup B_α and that all B_α are given as isomorphic to a group B . It is to be emphasized that there is a specific isomorphism given between each B_α and B . We wish to consider the most general group generated by the A_α in which all B_α are identified with each other so that all B_α form the same group B^1 isomorphic to B . This is clearly the image of the free product of the A_α

obtained by identifying in every case $b_\alpha \in B_\alpha$ and $b_\beta \in B_\beta$ if, in the given isomorphisms between B_α , B_β and B , b_α and b_β correspond to the same element b . The identification has essentially no effect on elements not in the B_α . The group generated by the A_α with all B_α 's identified with each other is called the amalgamated product of the A_α or the free product of the A_α with amalgamated subgroup B [3]. We denote the amalgamated product of H and G by $H * G$. We let $H * G \cdot U$ indicate that the amalgamated subgroups are isomorphic to U . Notice that $H * G \cdot Z$ means the amalgamated subgroups are infinite cyclic.

In the following discussion, a 2-sphere is a polyhedral 2-sphere unless otherwise specified. If S is a 2-sphere or torus (polyhedral), $\text{int } S$ and $\text{ext } S$ denote the bounded and unbounded components of $E^3 - S$. The closure of a set M is denoted by $\text{cl } M$. If S is a 2-sphere and A an arc in $\text{cl int } S$ meeting S in its end points only, we say that A is a spanning arc of S or that A spans S .

Definition 3.3: (A) Let S be a 2-sphere and A a spanning arc of S . Suppose $F = \pi[(\text{cl int } S) - A]$ has basepoint on S . A subgroup H of F is of type C_0 if and only if

- (i) each equivalence class of paths corresponding to an element of H has a path on S ,
- (ii) H is infinite cyclic and
- (iii) H is not a proper subgroup of a group with properties (i) and (ii).

(B) Let S be a torus and A a knot in $\text{int } S$. Then a subgroup H of $F = \pi[(\text{cl int } S) - A]$ is of type C_0 in the same manner as above.

Definition 3.4: If in Definition 3.3 S is a torus, then the group H will be called a longitudinal group if the nontrivial paths on S corresponding to elements of H are not contractible to a point in $\text{cl int } S$; otherwise, H is a latitudinal group.

Definition 3.5: Let $I = \{t \mid 0 \leq t \leq 1\}$ and $J = \{x \mid 0 \leq x \leq 1\}$. A is isotopic to B if there exists a family of maps $\{f_t\}$, t a member of I , such that $f_t(J)$ is a homeomorphism for each t and $f_0(J) = A$ and $f_1(J) = B$. We say that A is pseudoisotopic to B if the above maps exist except for f_1 which is continuous but not necessarily a homeomorphism.

Definition 3.6: A sequence of tori T_1, \dots, T_n will be called a t-sequence of tori for a knot K if and only if

- (i) T_1 is unknotted, i.e., $\pi(\text{ext } T_1) = Z$,
- (ii) $T_{i+1} \subset \text{int } T_i$, $1 \leq i \leq n$,
- (iii) there is a disk that meets each T_i in exactly one simple closed polygon,
- (iv) $\pi(\text{ext } T_i) = \pi(\text{ext } T_j)$ if and only if $i = j$,
- (v) T_{i+1} is not contractible to a point in $\text{int } T_i$ and
- (vi) T_n is pseudoisotopic to K .

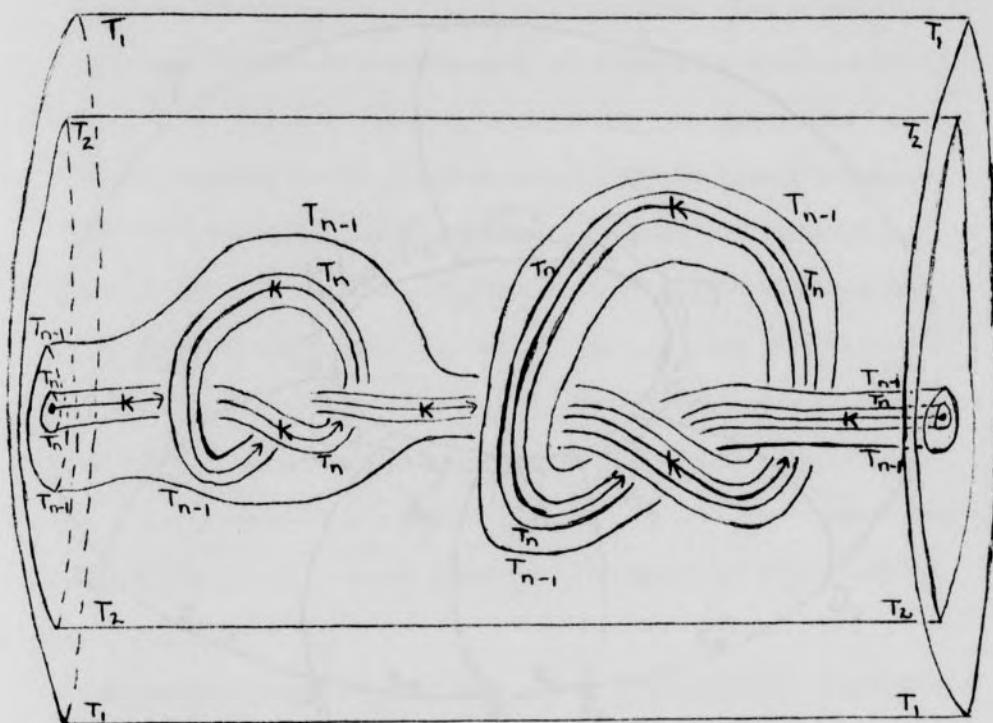


Figure 6

Section of a Tori Sequence for K

Definition 3.7: A sequence of 2-spheres S_1, \dots, S_n will be called a s-sequence of 2-spheres for a knot K if and only if

- (i) there is a torus T containing K in its interior and
- (ii) there are disks D_1, \dots, D_n spanning T and annuli E_1, \dots, E_n on T such that $S_i = D_i + E_i + D_{i+1}$, $n+1=1$, and $K = A_1 + \dots + A_n$ where A_i is an arc spanning S_i .

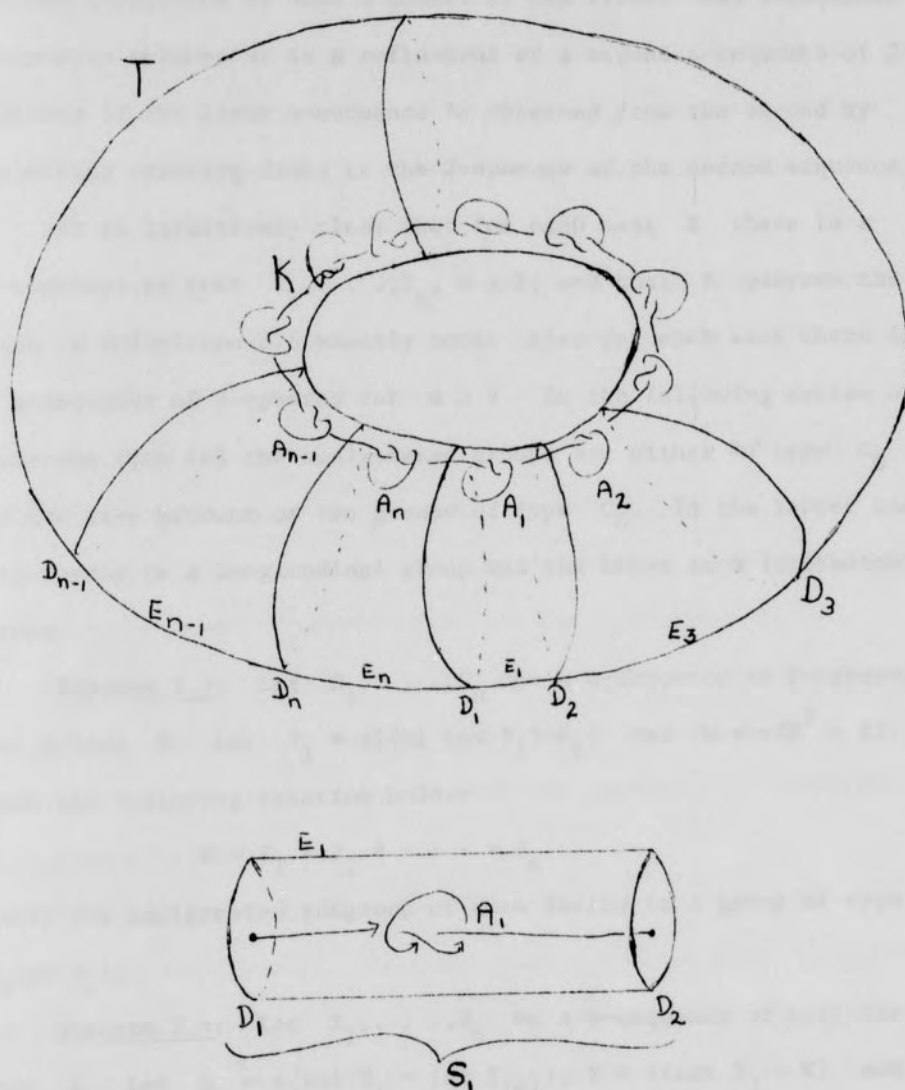


Figure 7

A s-sequence of 2-spheres for K

Definition 3.8: One t-sequence of tori refines or is a refinement of a second t-sequence of tori if each torus in the

second t -sequence is also a member of the first. One s -sequence of 2-spheres refines or is a refinement of a second s -sequence of 2-spheres if the first s -sequence is obtained from the second by adjoining spanning disks to the 2-spheres of the second sequence.

It is intuitively clear that for each knot K there is a t -sequence or tori T_1, \dots, T_n , $n \geq 2$, and that K pierces the disk in Definition 3.6 exactly once. Also for each knot there is a s -sequence of 2-spheres for $n \geq 2$. In the following series of theorems from [4] the amalgamated groups are either of type C_0 or the free product of two groups of type C_0 . In the latter case one factor is a longitudinal group and the other is a latitudinal group.

Theorem 3.3: Let S_1, \dots, S_n be a s -sequence of 2-spheres for a knot K . Let $F_i = \pi[(\text{cl int } S_i) - A_i]$ and $N = \pi(E^3 - K)$. Then the following relation holds:

$$N = F_1 * F_2 * \dots * F_n$$

where the amalgamated subgroup of each factor is a group of type C_0 (on S_i).

Theorem 3.4: Let T_1, \dots, T_n be a t -sequence of tori for a knot K . Let $H_i = \pi(\text{int } T_i - \text{int } T_{i+1})$, $F = \pi(\text{int } T_1 - K)$ and $N = \pi(E^3 - K)$. Then the following relations hold:

(i) $F = [H_1/Z_1 * \dots * H_n/Z_n] \otimes Z_1$ where Z_1 is a longitudinal subgroup corresponding to T_1 and the amalgamated subgroups are latitudinal and

(ii) $N = F/Z_1$.

Theorem 3.5: For each knot K the sequence of 2-spheres S_1, \dots, S_n and groups F_1, \dots, F_n in Theorem 3.3 exist if and only if the sequence of tori T_1, \dots, T_n and groups H_1, \dots, H_n in Theorem 3.4 exist. The groups may be indexed so that F_i is isomorphic to H_i/Z_i .

Theorem 3.6: For each knot there is a unique maximal number $N \geq 2$ such that each t -sequence and each s -sequence can be refined to a sequence with N elements.

The significance of Theorems 3.3, 3.4, 3.5 and 3.6 is that a knot may be factored in a systematic manner into parts. The fundamental group of each part may be calculated, then by amalgamating these groups together we obtain the knot group of the original knot. Also the t -sequence and the s -sequence for a knot are unique. Individual members of the sequences may be in a different order, but for a given knot the same tori or 2-spheres will always be mentioned.

We now illustrate the use of Theorems 3.3, 3.4, 3.5 and 3.6 in computing knot groups. We start with a trefoil knot K which has knot group $(x_1x_3 : x_1x_3x_1 = x_3x_1x_3)$ as shown in Chapter II. Next we attach an overhand knot, another name for a trefoil knot, to one arc of the original trefoil to obtain knot K_1 .

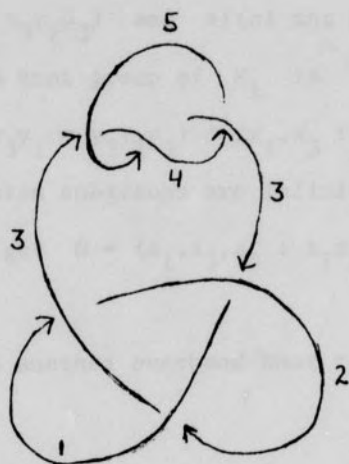


Figure 8

Construction of Knot K_1

We now employ Theorem 3.3 to calculate the knot group. The torus T has center as shown in Figure 9.

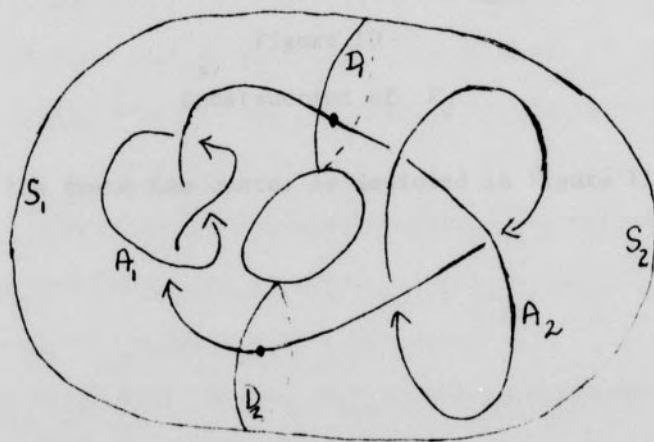


Figure 9

The 2-spheres for K_1

Since D_1 and D_2 may be identified, $\pi[(\text{cl int } S_1) - A_1] = (y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3)$ and $\pi[(\text{cl int } S_2) - A_2] = (x_1, x_3 : x_1 x_3 x_1 = x_3 x_1 x_3)$. Thus the knot group of K_1 is

$$G = (y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3) * (x_1, x_3 : x_1 x_3 x_1 = x_3 x_1 x_3) \cdot \mathbb{Z}$$

where the amalgamated subgroups are infinite cyclic. Simplifying and relabeling we get $G = (z_1, z_3, z_5 : z_1 z_3 z_1 = z_3 z_1 z_3, z_5 z_3 z_5 = z_3 z_5 z_3)$.

Now we attach another overhand knot to an arc of K_1 to obtain K_2 .

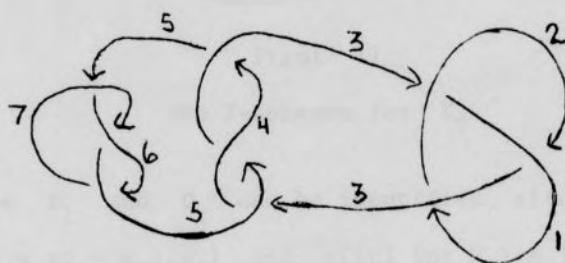


Figure 10

Construction of K_2

This time the torus has center as depicted in Figure 11.

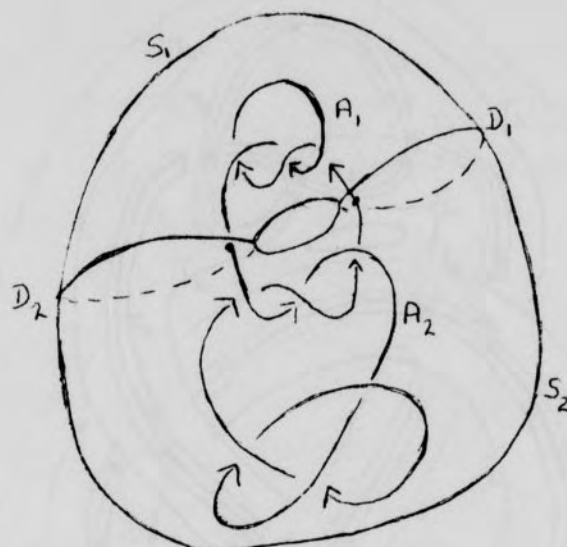


Figure 11

The 2-spheres for K_2

Again since D_1 and D_2 can be identified, $\pi[(\text{cl int } S_1) - A_1] = (y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3)$ and $\pi[(\text{cl int } S_2) - A_2] = (z_1, z_3, z_5 : z_1 z_3 z_1 = z_3 z_1 z_3, z_5 z_3 z_5 = z_3 z_5 z_3)$, the knot group of K_1 . Applying Theorem 3.3 once more we have H the fundamental group of K_3 is $(y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3) * (z_1, z_3, z_5 : z_1 z_3 z_1 = z_3 z_1 z_3, z_5 z_3 z_5 = z_3 z_5 z_3) \cdot \mathbb{Z}$ where the amalgamated subgroups are infinite cyclic. Simplifying and relabeling we obtain $(w_1, w_3, w_5, w_7 : w_1 w_3 w_1 = w_3 w_1 w_3, w_5 w_3 w_5 = w_3 w_5 w_3, w_5 w_7 w_5 = w_7 w_5 w_7)$.

We now start with K and K_1 in the same manner as above and illustrate Theorem 3.4. Let T_1, T_2 and T_3 be tori as depicted in Figure 12.

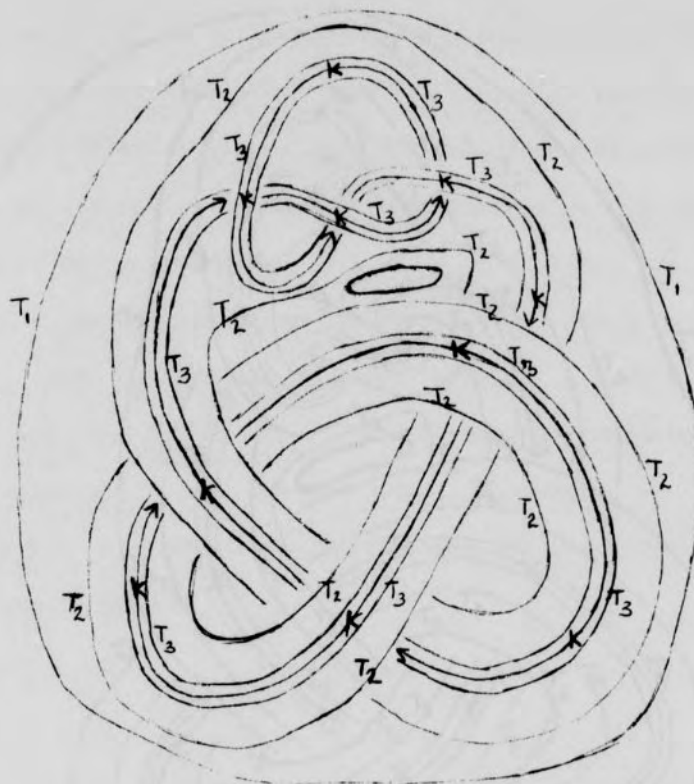


Figure 12

Tori Sequence for K_1

Following the notation in Theorem 3.4, we have $H_1 = \pi(\text{int } T_1 - \text{int } T_2) = (x_1, x_3 : x_1 x_3 x_1 = x_3 x_1 x_3)$ the knot group of the trefoil knot. Likewise $H_2 = \pi(\text{int } T_2 - \text{int } T_3) = (y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3)$. Hence $F = [H_1 / (Z_1) * H_2 / (Z_2)] \otimes Z_1$ or $N = (z_1, z_3, z_5 : z_1 z_3 z_1 = z_3 z_1 z_3, z_5 z_3 z_5 = z_3 z_5 z_3)$ what we found to be the knot group G of K_1 . For K_2 we let T_1, T_2, T_3 and T_4 be as pictured in Figure 13.

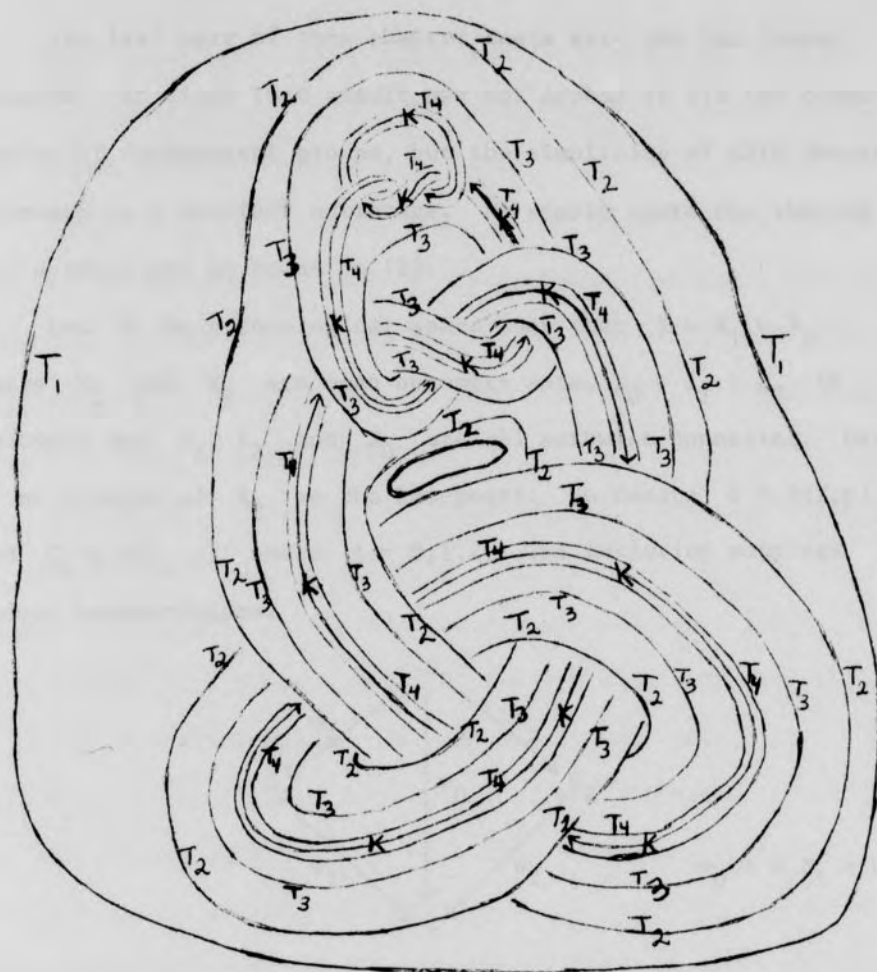


Figure 13

Tori sequence for K_2

Once more we have $H_1 = \pi(\text{int } T_1 - \text{int } T_2) = (x_1, x_3 : x_1 x_3 x_1 = x_3 x_1 x_3)$, $H_2 = \pi(\text{int } T_2 - \text{int } T_3) = (y_1, y_3 : y_1 y_3 y_1 = y_3 y_1 y_3)$ and $H_3 = \pi(\text{int } T_3 - \text{int } T_4) = (z_1, z_3 : z_1 z_3 z_1 = z_3 z_1 z_3)$. Theorem 3.4 yields $F = [H_1/(Z_1) * H_2/(Z_2) * H_3/Z_3] \otimes Z_1$ and the knot group N of K_2 is $(w_1, w_3, w_5, w_7 : w_1 w_3 w_1 = w_3 w_1 w_3, w_5 w_3 w_5 = w_3 w_5 w_3, w_5 w_7 w_5 = w_7 w_5 w_7)$ as above.

The last part of this chapter deals with the van Kampen Theorem. At first this result may not appear to aid the computation of fundamental groups, but the simplicity of this abstract approach is a distinct advantage. We simply state the theorem but a proof may be found in [1].

Let X be a topological space such that $X = X_1 \cup X_2$ where X_1 and X_2 are open nonempty sets, $X_0 = X_1 \cap X_2$ is nonempty and X_1 , X_2 and X_0 are all pathwise connected. Let p an element of X_0 be the basepoint. We denote $G = \pi(X, p)$ and $G_i = \pi(X_i, p)$ where $i = 0, 1, 2$. The inclusion mappings induce homomorphisms.

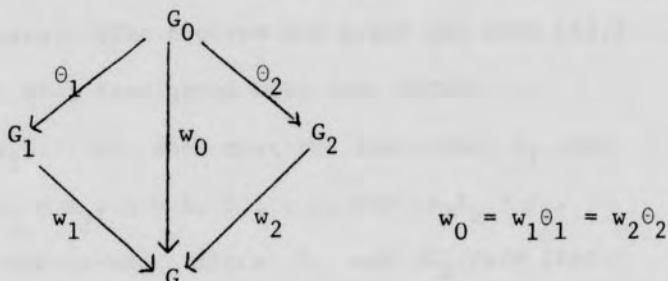


Figure 14

Homomorphisms for the van Kampen Theorem

Theorem 3.7 (The van Kampen Theorem): The image groups $w_i G_i$, $i = 0, 1, 2$, generate G . Furthermore, if H is an arbitrary group and $\psi_i : G_i \rightarrow H$, $i = 0, 1, 2$, are homomorphisms which satisfy $\psi_0 = \psi_1 \theta_1 = \psi_2 \theta_2$, then there exists a unique homomorphism $\lambda : G \rightarrow H$ such that $\psi_i = \lambda w_i$, $i = 0, 1, 2$.

Suppose we wish to calculate the fundamental group of two circles joined at one point. The next theorem, a result of the van Kampen Theorem, solves our problem.

Definition 3.9: Let G be a group. Let E be a generating set of elements of G . Then E is a free basis for G if, given any group H , any function $\phi : E \rightarrow H$ can be extended to a homomorphism of G into H . A group that has a free basis will be called free or a free group.

Theorem 3.8: If G_0 is trivial and G_1 and G_2 are free groups with free bases $\{\alpha_1, \alpha_2, \dots\}$ and $\{\beta_1, \beta_2, \dots\}$, respectively, then G is free and $\{w_1\alpha_1, w_1\alpha_2, \dots, w_2\beta_1, w_2\beta_2, \dots\}$ is a free basis. (The theorem and proof are from [1].)

Proof: Let H be a free group with free basis $\{x_1, x_2, \dots, y_1, y_2, \dots\}$ such that the functions ψ_1 and ψ_2 defined by $\psi_1 \alpha_j = x_j$, $j = 1, 2, \dots$, and $\psi_2 \beta_k = y_k$, $k = 1, 2, \dots$, are one-to-one. Since G_1 and G_2 are free, these correspondences extend to homomorphisms $\psi_i : G_i \rightarrow H$, $i = 1, 2$. Since G_0 is trivial, there is a trivial homomorphism $\psi_0 : G_0 \rightarrow H$ and $\psi_0 = \psi_1 \circ \theta_1 = \psi_2 \circ \theta_2$. By Theorem 3.7, there exists a homomorphism $\lambda : G \rightarrow H$ such that $\psi_i = \lambda w_i$, $i = 0, 1, 2$. Hence $\lambda w_1 \alpha_j = \psi_1 \alpha_j = x_j$, $j = 1, 2, \dots$, and $\lambda w_2 \beta_k = \psi_2 \beta_k = y_k$, $k = 1, 2, \dots$. Since H is free, there exists a homomorphism $\mu : H \rightarrow G$ defined by $\mu x_j = w_1 \alpha_j$ and $\mu y_k = w_2 \beta_k$. Obviously, both $\lambda \mu$ and $\mu \lambda$ are identity mappings. Hence, both isomorphisms are onto and inverses of each other.

Definition 3.10: A retraction of a topological space X onto a subspace Y is a continuous mapping $f : X \rightarrow Y$ such that, for any p in Y , $f(p) = p$.

Definition 3.11: A deformation of a topological space X is a family of mappings $h_s : X \rightarrow X$, $0 \leq s \leq 1$, such that h_0 is the identity and the function h defined by $h(s, p) = h_s(p)$ is simultaneously continuous in two variables s and p . An arbitrary continuous mapping $f : X \rightarrow Y$ of a topological space X into a subspace Y is said to be realizable by a deformation of X if there exists a deformation $\{h_s\}$, $0 \leq s \leq 1$, of X such that $h_1 = f \circ i$ where $i : Y \rightarrow X$ is the inclusion mapping.

Definition 3.12: A subspace Y of a topological space X is a deformation retract of X if there exists a retraction $G : X \rightarrow Y$ which is realizable by a deformation of X .

We now calculate the fundamental group of the n -leafed rose, denoted by $C(n)$, which is the union of n topological circles X_1, \dots, X_n joined at a single point p and otherwise disjoint. Notice the two circles with one point in common is a 2-leafed rose. We will show the fundamental group of $C(n)$ is free of rank n (the number of elements in the free basis). In other words, if x_i is a generator of the infinite cyclic group X_i and $w_i : \pi(X_i) \rightarrow \pi(C(n))$ is induced by the inclusion map, then $\pi(C(n)) = \langle w_1 x_1, \dots, w_n x_n \rangle$. (If the group is free we omit relations from our group presentations.) We perform induction on n . We know from Theorem 1.4 that the space $C(1)$, a circle, is infinite cyclic or

free of rank 1. Suppose $C(n+1) = C(n) \cup X_{n+1}$ and $\{p\} = C_n \cap X_{n+1}$. Our conclusion could follow immediately from Theorem 3.8 except that $C(n)$, X_{n+1} and $\{p\}$ are not open in $C(n+1)$. Let N be an open neighborhood of p in $C(n+1)$ consisting of p and the union of $2(n+1)$ disjoint, open arcs with p as one end point of each arc. Then $C(n)$, X_{n+1} , and $\{p\}$ are deformation retracts of $C(n) \cup N$, $X_{n+1} \cup N$, and $\{p\} \cup N$, respectively. The latter are open subsets of $C(n+1)$ and we now apply Theorem 3.8 to complete our induction. Hence the fundamental group of 2 circles joined at a point is the free group of rank 2.

The final objective of this chapter is to calculate the fundamental group of the torus. We divide the torus X in the following way: X_1 is the torus minus a closed disk D and X_2 is an open disk which contains D . Then $X_0 = X_1 \cap X_2$ is an open annulus which can be shown to have an infinite cyclic fundamental group. The space X_1 is of the same homotopy type as the 2-leafed rose which can be geometrically shown by stretching D .

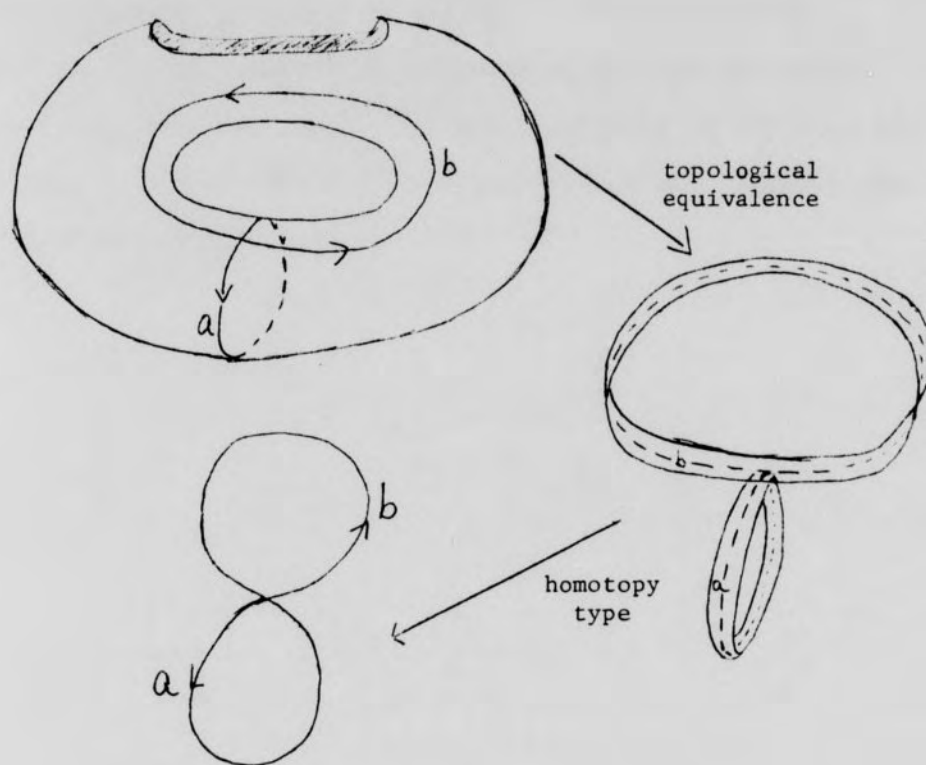


Figure 15

Homotopy Type of X_1

Hence $\pi(X_1)$ is free of rank 2. It can be shown that a generator of $\pi(X_0)$ is represented by a path c running around the edge of X_1 and X_2 . From the figure it is geometrically clear that c is equivalent in X_1 to $aba^{-1}b^{-1}$. Hence $[c] = [a][b][a^{-1}][b^{-1}]$. Consequently, $\pi(X_1)$ is a free group, $x = [a]$ and $y = [b]$ constitute a free basis, X_2 is simply connected, and $\pi(X_0)$ is generated by an element whose image in $\pi(X_1)$ under the homomor-

phism induced by inclusion is $xyx^{-1}y^{-1}$. The homomorphism $w_1 : \pi(X_1) \rightarrow \pi(X)$ induced by inclusion is onto and its kernel is the consequence of $xyx^{-1}y^{-1}$. Hence the group of the torus has presentation $(x, y : xyx^{-1}y^{-1})$ or $(x, y : xy = yx)$. This is the free abelian group of rank 2.

SUMMARY

There are still a lot of unanswered questions about knots and knot groups. We will mention a few. Every knot has a group but does every group have a knot? If not, what properties does a group need to have a knot associated with it?

It was shown in Chapter II that two distinct (unequivalent) knot types may have the same knot group. Can nongroup properties be associated with the group so that the group will completely characterize the knot? It is Dr. E. E. Posey's conjecture that if an isomorphism exists between G_1 , the knot group of K_1 , and G_2 , the knot group of K_2 , such that the longitudinal elements of G_1 are mapped onto the longitudinal elements of G_2 and the latitudinal elements of G_1 are mapped onto the latitudinal elements of G_2 , then K_1 is equivalent to K_2 . A correspondence between the two t -sequences of tori for K_1 and K_2 determines a correspondence between the longitudinal elements of G_1 and the longitudinal elements of G_2 and a correspondence between the two s -sequences of 2-spheres for K_1 and K_2 determines a correspondence between the latitudinal elements of G_1 and the latitudinal elements of G_2 .

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